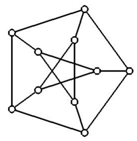

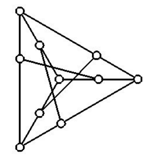

Research Statement

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My mathematical research is in the area of combinatorics, with an emphasis on algebraic graph theory. The process in which I study combinatorial objects involves applying modern experimental techniques using state of the art computer algebra packages, like Magma and GAP, to find promising evidence for conjectures of interest. I then go on to prove these conjectures.

The underlying techniques that I use are very general and involve applying symmetry hypotheses. In order to apply these techniques, the combinatorial object that I want to construct must have an underlying group. Thus, I make a weak symmetry hypothesis about small subgroups of the ambient group, not assumed to be acting nicely, and seek interesting outcomes experimentally. The experimental process, as well as theory, leads to the hypotheses. The hypotheses are not so much *a priori* as informed by a cycle of experimentation and theory. This technique allows for the rapid abandonment of unpromising lines of thinking; it is known in the literature as the spiral approach. Experiment feeds back into theory that sharpens algorithms. This, in turn, allows for more experimentation, leading to a spiraling process. In particular, the techniques I use are effective across a broad range of problems, which gives me versatility. These techniques can also be used for undergraduate research projects.

I. Research Synopsis

As previously mentioned, my area of study is algebraic graph theory. More specifically, I construct new strongly regular graphs. A *strongly regular graph* is a graph with constant numbers of neighbors of: a vertex, an edge, and a pair of vertices that are not joined. These constants, together with the number of vertices, are called the parameters of the graph. For example, the graph with vertices the subsets of size two of a set of size 5, joined whenever they are disjoint is a famous strongly regular graph called the Petersen graph: it has 10 vertices; each vertex has 3 neighbors; each edge has 0 neighbors, and each pair of vertices that are not joined has 1 common neighbor. In summary, it has parameters (10,3,0,1). Two different images of the Petersen graph can be found above.

I utilize two different symmetry techniques when constructing new strongly regular graphs. Nevertheless, both techniques follow the same experimental procedure that was previously outlined and the underlying connection between all results boils down to finite geometry. Furthermore, all the graphs I have constructed have much larger parameters than that of the Petersen graph.

The first approach exploits the symmetry hypothesis by applying finite vector spaces. In this case, the underlying graph is the Cayley graph. This means that there is a finite vector space V that can be identified with the vertices, and the neighbors N of the zero vector determine the graph. Because the graph is undirected, N is closed under negatives. I can then strengthen this hypothesis to N being closed under all non-zero scalars, so that N determines a set X of points in the projective space PV. Through the creation of X, the graph $\Gamma(X)$ is defined. The vertices of $\Gamma(X)$ are V, and two points in V, say v and w, are adjacent if, and only if, $< v - w >\in X$. This procedure is reflective of basic undergraduate linear algebra. The issue now is determining whether or not $\Gamma(X)$ is a strongly regular graph. The resulting graph will be strongly regular if, and only if, the set of neighbors of the zero vector is the union of a set X of points of the corresponding projective space PV which has two intersection sizes with hyperplanes. This is due to a theorem of Delsarte from 1968 (which also gives connections to error-correcting codes in this case) [1].

In this situation, I have constructed a number of strongly regular graphs with new parameters by finding appropriate further symmetry hypotheses. For instance, in the case where V is three-dimensional and the underlying field has order q^2, assuming that $\mathrm{PSL}(2, \mathrm{q})$ is admitted, has been fruitful. This approach has led to nine new strongly regular graphs with six of them corresponding to graphs with previously unknown parameters [7]. Under these hypotheses, I have also constructed an infinite family with the same parameters as the Paley graphs when V is of dimension six [5].

An alternative fruitful symmetry hypothesis has been to construct graphs with the same parameters as previously known strongly regular graphs Γ admitting classical groups. The new graphs have smaller automorphism groups than that of Γ, which are also subgroups of the classical groups in question. The idea is that symmetry replaces edges. For non-linear classical groups of Lie rank two, these graphs Γ are the point graphs of generalized quadrangles. A *generalized quadrangle* is a bipartite graph of diameter four and girth eight. For a graph to be bipartite, its vertices must be divided into two disjoint sets such that every edge connects a vertex in one set to a vertex in the other. The girth of a graph is the length of the shortest cycle contained in the graph, while the diameter is the largest number of vertices which must be traversed in order to travel from one vertex to another. Using a particular family of generalized quadrangles, a new family of strongly regular graphs has been constructed [6].

Generalized quadrangles turn up in my research in two other ways: generalizing a construction of Godsil and Hensel from finite geometric objects called ovoids of generalized quadrangles to give more strongly regular graphs [2], and using ovals (another object in finite geometry) to construct infinite families of strongly regular graphs with the same parameters as point graphs of strongly regular graphs, in joint work with Stan Payne [3]. Thus a number of concepts from finite geometry (such as two intersection sets, generalized quadrangles, ovoids and ovals) underlie many of my constructions. However, this is inessential. The methods work without the underlying geometry. It is only a useful conceptual scaffolding.

2. Future Work

Using the previously mentioned techniques, I have obtained numerous leads on new strongly regular graphs, which still need to be analyzed [4]. Ideally, the analysis will lead to more infinite families. In addition to that, I need to prove that the constructions found using generalized quadrangles does, in fact, lead to new infinite families of strongly regular graphs. I would also like to consider other groups for known strongly regular graphs to see if I can obtain new graphs.

In addition to these projects, I am extremely interested in working on projects with undergraduate students. A major benefit of the methods I use, is that they are widely applicable in many areas of combinatorics. For example, these techniques can be applied to codes, designs, and distance-regular graphs. These objects, as well as many other combinatorial objects, consist of very little overhead. This translates well to undergraduate projects. I especially think that projects on codes have the potential to spark a lot of undergraduate interest, as both pure and applied mathematicians find them interesting. This is definitely an avenue I would like to explore.

Furthermore, my master's research dealt with Ramsey theory, specifically finding 2-color disjunctive Rado numbers for sets of equations [8]. This is an area I would like to explore in further detail as well. These problems also translate well to undergraduate projects. Stemming from my master's work, I would like to investigate the 3-color disjunctive Rado number for the same set of equations. I can also alter one equation by adding a constant and investigate that. I think that working on projects like these with undergraduate students is feasible and potentially fruitful.

Lastly, I would like to conduct educational research as well. While teaching at Black Hills State University, we implemented a new course design for the remedial math courses, which significantly increased pass rates without compromising the material. I am inclined to pursue course development and implement changes, as necessary, at other universities. While completely overhauling a course is not always feasible, making small changes within a course can also lead to positive outcomes. While at Colorado State University, I, along with two professors and another graduate student, analyzed, with the aid of statistical software, the types of students enrolling in Calculus for Physical Sciences I. One of the goals of the analysis was to help instructors pinpoint areas of student weakness. For example, if I can determine whether or not students are coming in with a sufficient algebraic background, I can tailor my lectures accordingly. I would like to continue with such analysis, solidifying positive and effective teaching techniques within the realm of mathematics.

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